The exactly solvable quasi-Hermitian transverse Ising model

Tetsuo Deguchi[1] **and Pijush K Ghosh**[2]

[1] Department of Physics, Graduate School of Humanities and Sciences, Ochanomizu University, 2-1-1 Ohtsuka, Bunkyo-ku, Tokyo 112-8610, Japan
[2] Department of Physics, Siksha-Bhavana, Visva-Bharati University, Santiniketan 731 235, West Bengal, India

E-mail: deguchi@phys.ocha.ac.jp and pijushkanti.ghosh@visva-bharati.ac.in

Abstract

A non-Hermitian deformation of the one-dimensional transverse Ising model is shown to have the property of quasi-hermiticity. The transverse Ising chain is obtained from the starting non-Hermitian Hamiltonian through a similarity transformation. Consequently, both the models have identical eigen spectra, although the eigenfunctions are different. The metric in the Hilbert space, which makes the non-Hermitian model unitary and ensures the completeness of states, has been constructed explicitly. Although the longitudinal correlation functions are identical for both the non-Hermitian and the Hermitian Ising models, the difference shows up in the transverse correlation functions, which have been calculated explicitly and are not always real. A proper set of Hermitian spin operators in the Hilbert space of the non-Hermitian Hamiltonian has been identified, in terms of which all the correlation functions of the non-Hermitian Hamiltonian become real and identical to that of the standard transverse Ising model. Comments on the quantum phase transitions in the non-Hermitian model have been made.

PACS numbers: 02.30.Ik, 75.10.Pq, 03.65.Fd

The discovery of a class of non-Hermitian Hamiltonians admitting entirely real spectra has generated a renewed interest in the study of quantum physics [1–10]. The reality of the entire spectra is related to an underlying unbroken combined parity ($\mathcal{P}$) and time-reversal ($\mathcal{T}$) symmetry [1] and/or quasi-hermiticity [2, 3] of the non-Hermitian Hamiltonian. Apart from a very few known examples [6–9], one of the major technical difficulties in the study of $\mathcal{PT}$ symmetric and/or quasi-Hermitian quantum physics is to find the appropriate basis with respect to which the non-Hermitian Hamiltonian becomes Hermitian. The description of a non-Hermitian Hamiltonian admitting entirely real spectra is incomplete in the absence of such a basis, since neither the unitarity nor the completeness of states is guaranteed. It may

be noted here that the completeness of states is an essential criterion to claim a Hamiltonian to be exactly solvable.

The purpose of this paper is to introduce and study an exactly solvable non-Hermitian Hamiltonian of the type of the transverse Ising model that admits entirely real spectra. In particular, we consider a non-Hermitian Hamiltonian and map it to the transverse Ising model through a similarity transformation. Consequently, both the models have identical spectra. We find the metric in the Hilbert space of the non-Hermitian Hamiltonian that is required to make the theory unitary and to ensure the completeness of states. We show that the n-point longitudinal correlation function of the non-Hermitian Hamiltonian is identical to that of the standard Hermitian transverse Ising model. We also calculate the two-point transverse correlation functions of the non-Hermitian Hamiltonian exactly that reduces to that of the transverse Ising model in the Hermitian limit. However, the transverse correlation functions are not always real. We identify a proper set of Hermitian spin operators in the Hilbert space of the non-Hermitian Hamiltonian in terms of which all the correlation functions of the non-Hermitian Hamiltonian become real and identical to that of the standard transverse Ising model.

There are many physical applications of the transverse Ising chain such as quantum phase transitions and finite-temperature crossovers [11–13]. The transverse Ising model has also been studied [14] extensively from the viewpoint of quantum entanglement and its connection with quantum phase transition. We may thus expect that the non-Hermitian transverse Ising Hamiltonian gives an explicit and concrete example of a non-Hermitian exactly solvable many-body system and should be useful for studying some interesting properties of non-Hermitian quantum systems explicitly.

Non-Hermitian quantum many-body systems are closely related to several important topics in other subjects. For instance, non-Hermitian quantum spin chains correspond to two-dimensional classical systems with positive Boltzmann weights. In exactly solvable models, the non-Hermitian XY and XXZ spin chain Hamiltonians with Dzyaloshinsky–Moriya interactions commute with the transfer matrix of the six-vertex model in the presence of an electric field [15], and the integrable chiral Potts model in the most general case leads to a non-Hermitian quantum Hamiltonian (see for review [16, 17]). Moreover, non-Hermitian asymmetric XXZ spin chains related to the one-dimensional diffusion models have been studied extensively in nonequilibrium statistical mechanics [18]. The inherent pseudo-hermiticity of these spin models has been discovered very recently [19] which allows a unitary description with a modified inner product in the Hilbert space. Further, a non-Hermitian quantum Ising spin chain in one dimension [20] is known to be related to the celebrated Yang–Lee model [21] that aptly describes ordinary second-order phase transitions. The non-hermiticity of the spin chain arises due to the inclusion of an external complex magnetic field and an analysis based on minimal conformal field theories is available [20, 22]. Very recently, pseudo-hermiticity of the non-Hermitian Ising chain of [20] has been studied for a finite number of sites in [23] and any result for an arbitrary number of sites is still lacking. The non-Hermitian quantum Ising chain that is considered in this paper is different from that of [20] and an exact description of an arbitrary number of sites is possible.

Let us now consider the transverse Ising Hamiltonian in the following modified form,

$$H = -\sum_{i=1}^{N} \left(J S_i^z S_{i+1}^z + \epsilon_1 S_i^+ + \epsilon_2 S_i^- \right) \tag{1}$$

where $S_i^z, S_i^\pm = \left(S_i^x \pm i S_i^y \right)$ are the spin variables. The spin variables can be represented in terms of the Pauli matrices $\sigma^\pm = \frac{1}{2} \left(\sigma^x \pm i \sigma^y \right), \sigma^z$ and the 2×2 identity matrix I as

$$S_i^z = I \otimes \cdots \otimes I \otimes \tfrac{1}{2} \sigma^z \otimes I \otimes \cdots \otimes I$$
$$S_i^\pm = I \otimes \cdots \otimes I \otimes \sigma^\pm \otimes I \otimes \cdots \otimes I, \tag{2}$$

where $\sigma^{\pm}$ and σ^z are in the ith position. The parameter J is real, however, $\epsilon_{1,2}$ are complex. Thus, the Hamiltonian is non-Hermitian for $\epsilon_1 \neq \epsilon_2^*$, where the $*$ denotes the complex conjugation. The Hamiltonian H and its adjoint $H^{\dagger}$ are related to each other through the transformation $\epsilon_1 \leftrightarrow \epsilon_2^*$. The standard transverse Ising model is recovered when both ϵ_1 and ϵ_2 are real and $\epsilon_1 = \epsilon_2$. Even for the Hermitian case, i.e. $\epsilon_1 = \epsilon_2^*$, H can be mapped to the standard transverse Ising model through a unitary transformation. However, for general ϵ_1 and ϵ_2, H cannot be mapped to a Hermitian Hamiltonian by using a unitary transformation.

The Hamiltonian H can be mapped to a Hermitian Hamiltonian through a similarity transformation. To do so, let us introduce the operator ρ and its inverse ρ^{-1} in the following way:

$$
\begin{aligned}
\rho_i &= \gamma^{-\frac{1}{2}} S_i^+ S_i^- + \gamma^{\frac{1}{2}} S_i^- S_i^+, \\
\rho_i^{-1} &= \gamma^{\frac{1}{2}} S_i^+ S_i^- + \gamma^{-\frac{1}{2}} S_i^- S_i^+, \\
\rho &= \prod_{i=1}^{N} \rho_i, \quad \rho^{-1} = \prod_{i=1}^{N} \rho_i^{-1},
\end{aligned} \tag{3}
$$

where $\gamma = \sqrt{\frac{|\epsilon_1|}{|\epsilon_2|}}$. The ordering of ρ_i's is not required in the definition of the positive-definite operators ρ and ρ^{-1}, since $[\rho_i, \rho_j] = 0$ for $i \neq j$. Using the following identities,

$$
\begin{aligned}
\rho \, S_i^z \, \rho^{-1} &= S_i^z, \\
\rho \, S_i^{\pm} \, \rho^{-1} &= \gamma^{\mp 1} \, S_i^{\pm},
\end{aligned} \tag{4}
$$

one can easily check that

$$
\begin{aligned}
h &= \rho H \rho^{-1} \\
&= -\sum_{i=1}^{N} J S_i^z S_{i+1}^z \\
&\quad - \beta \sum_{i=1}^{N} (e^{i \arg(\epsilon_1)} \, S_i^+ + e^{i \arg(\epsilon_2)} \, S_i^-).
\end{aligned} \tag{5}
$$

The parameter β appearing in h is defined as $\beta \equiv \sqrt{|\epsilon_1||\epsilon_2|}$. The Hamiltonian h is Hermitian when the following condition holds true:

$$
\arg(\epsilon_1) + \arg(\epsilon_2) = 2k\pi, \qquad k = 0, 1, 2, \dots. \tag{6}
$$

Thus, equation (6) is the condition for H to be quasi-Hermitian, i.e. related to the Hermitian h through the similarity transformation. A counter-clockwise rotation on the S_x–S_y-plane around the S_z-axis by an angle $\xi \equiv \arg(\epsilon_1) = -\arg(\epsilon_2)$, followed by a clockwise rotation by an angle $\frac{\pi}{2}$ around the S_y-axis for each spin transform h into the standard form of the transverse Ising model. To this end, we introduce an operator U as

$$
U = \prod_{i=1}^{N} e^{i \frac{\pi}{2} S_i^y} \prod_{j=1}^{N} e^{-i\xi S_j^z}, \tag{7}
$$

which transforms $S_i^{x,y,z}$ in the following way:

$$
\begin{aligned}
U S_i^z U^{-1} &= -S_i^x, \\
U S_i^y U^{-1} &= S_i^y \cos\xi - S_i^z \sin\xi, \\
U S_i^x U^{-1} &= S_i^z \cos\xi + S_i^y \sin\xi.
\end{aligned} \tag{8}
$$

3

Using the above identities, h can be transformed to $\mathcal{H}$,

$$\mathcal{H} = U h U^{-1}$$
$$= -\sum_{i=1}^{N} \left(J S_i^x S_{i+1}^x + \beta S_i^z \right) \tag{9}$$

which is the standard form of the transverse Ising model. The entire energy spectra of H are real and identical to that of $\mathcal{H}$, since a similarity transformation cannot change the eigenvalues. However, as we will see below, the difference between H and the transverse Ising Model (i.e. H with real $\epsilon_1 = \epsilon_2$) shows up in the eigenstates and the transverse correlation functions.

The transverse Ising model $\mathcal{H}$ is exactly solvable and the different correlation functions can be calculated explicitly [24–27]. Using the Jordan–Wigner transformation, the Hamiltonian $\mathcal{H}$ can be transformed to a fermionic Hamiltonian which is quadratic in the fermionic annihilation and creation operators. The resulting fermionic Hamiltonian can be further diagonalized in terms of a new set of canonical Fermi operators [24]. It is worth mentioning here that the direct application of the Jordan–Wigner transformation to H produces a fermionic Hamiltonian with non-local, non-Hermitian interaction. However, the operators ρ and its inverse ρ^{-1} defined appropriately in terms of fermionic annihilation and creation operators transform H to the fermionic version of $\mathcal{H}$ that is local and Hermitian.

If $|\psi_n\rangle$ constitutes a complete set of orthonormal eigenstates of the Hermitian Hamiltonian $\mathcal{H}$ with energy eigenvalue E_n, then

$$|\phi_n\rangle = (U \rho)^{-1} |\psi_n\rangle, \qquad |\chi_n\rangle = (\rho \, U^{-1}) |\psi_n\rangle \tag{10}$$

are the eigenstates of H and its adjoint $H^\dagger$, respectively. It may be noted here that both H and $H^\dagger$ share the same energy eigenvalue E_n with $\mathcal{H}$. However, neither $|\phi_n\rangle$ nor $|\chi_n\rangle$ constitute a complete set of orthonormal basis vectors. Consequently, with the standard norm in the Hilbert space, the time evolution of H(or $H^\dagger$) is not unitary, although the entire eigen spectra are real. As is evident from equations (5)–(9), H is a quasi-Hermitian operator. Thus, the Hilbert space of H admits a bi-orthogonal structure,

$$\langle \chi_n | \phi_m \rangle = \delta_{nm}, \qquad \sum_n |\chi_n\rangle\langle\phi_n| = 1. \tag{11}$$

The completeness of states can be accomplished if the inner-product in the Hilbert space is modified as [2]

$$\langle\langle u, v \rangle\rangle_{\eta_+} := \langle u, \eta_+ v \rangle, \qquad \eta_+ := \rho^2. \tag{12}$$

With this new inner-product in the Hilbert space, the expectation value of an operator $\hat{O}$ can be calculated as

$$\langle\langle \hat{O} \rangle\rangle_{\eta_+} \equiv \langle \phi_n | \eta_+ \hat{O} | \phi_n \rangle = \langle \psi_n | (U\rho) \, \hat{O} \, (U\rho)^{-1} | \psi_n \rangle. \tag{13}$$

We will be using the above expression to calculate n-point correlation function of H. The standard inner product $\langle u, v \rangle$ will be used to calculate the correlation function of the Hermitian Hamiltonian $\mathcal{H}$.

An n-point ($n \leqslant N$) longitudinal correlation function of H for the mth eigenstate can be related to the correlation function of the transverse Ising model in the following way:

$$\left\langle\left\langle S_{i_1}^z S_{i_2}^z \cdots S_{i_n}^z \right\rangle\right\rangle_{\eta_+} = (-1)^n \langle \psi_m | S_{i_1}^x S_{i_2}^x \cdots S_{i_n}^x | \psi_m \rangle, \tag{14}$$

where any two of the indices i_k are not equal. Identifying S_i^z of H with $-S_i^x$ of $\mathcal{H}$, we observe that the longitudinal correlation functions for these two systems are identical. However, the n-point transverse correlation functions of H and $\mathcal{H}$ differ from each other. Let us introduce a

complex parameter $z \equiv \ln \gamma + i\xi$, $\gamma > 0$ in terms of γ and ξ. We also introduce two operators $Q_{i_1,i_2,\ldots,i_n}$ and $\tilde{Q}_{i_1,i_2,\ldots i_n}$ as

$$Q_{i_1 i_2 \cdots i_n} = \prod_{j=1}^{n} \left(\cosh z\, S_{i_j}^z - i \sinh z\, S_{i_j}^y \right)$$

$$\tilde{Q}_{i_1 i_2 \cdots i_n} = \prod_{j=1}^{n} \left(i \sinh z\, S_{i_j}^z + \cosh z\, S_{i_j}^y \right). \tag{15}$$

The transverse correlation functions of H and $\mathcal{H}$ can now be related as

$$\left\langle\!\left\langle S_{i_1}^x S_{i_2}^x \cdots S_{i_n}^x \right\rangle\!\right\rangle_{\eta_+} = \langle \psi_m | Q_{i_1 i_2 \cdots i_n} | \psi_m \rangle$$

$$\left\langle\!\left\langle S_{i_1}^y S_{i_2}^y \cdots S_{i_n}^y \right\rangle\!\right\rangle_{\eta_+} = \langle \psi_m | \tilde{Q}_{i_1 i_2 \cdots i_n} | \psi_m \rangle. \tag{16}$$

In general, for $\gamma \neq 1$, the correlation functions are complex. For example, the one-point correlation functions in the ground-state $|\psi_0\rangle$ can be evaluated as

$$\left\langle\!\left\langle S_i^x \right\rangle\!\right\rangle_{\eta_+} = \cosh z\, M_i^z, \qquad \left\langle\!\left\langle S_i^y \right\rangle\!\right\rangle_{\eta_+} = i \sinh z\, M_i^z, \tag{17}$$

where $M_i^{x,y,z} \equiv \langle \psi_0 | S_i^{x,y,z} | \psi_0 \rangle$ and we have used the result [26] that $M_i^y = 0$ for arbitrary $\lambda \equiv \frac{J}{\beta}$. It may be noted that for $\gamma \neq 1$, $\left\langle\!\left\langle S_i^x \right\rangle\!\right\rangle_{\eta_+}$ is real only for $\xi = n\pi$,

$$\left\langle\!\left\langle S_i^x \right\rangle\!\right\rangle_{\eta_+} = (-1)^n \cosh(\ln \gamma)\, M_i^z, \tag{18}$$

while $\left\langle\!\left\langle S_i^y \right\rangle\!\right\rangle_{\eta_+}$ is real only for $\xi = (2n+1)\frac{\pi}{2}$,

$$\left\langle\!\left\langle S_i^y \right\rangle\!\right\rangle_{\eta_+} = (-1)^{n+1} \cosh(\ln \gamma)\, M_i^z, \tag{19}$$

where n is either zero or a positive integer. It is expected that for H to describe a physical theory, at least both the magnetization along the X- and Y-directions should be real, which is not the case for a fixed ξ and $\gamma \neq 1$. This is certainly an unwanted feature.

The two-point diagonal correlation functions have the following form:

$$\left\langle\!\left\langle S_i^x S_j^x \right\rangle\!\right\rangle_{\eta_+} = \cosh^2 z\, C_{ij}^z - \sinh^2 z\, C_{ij}^y,$$

$$\left\langle\!\left\langle S_i^y S_j^y \right\rangle\!\right\rangle_{\eta_+} = - \sinh^2 z\, C_{ij}^z, + \cosh^2 z\, C_{ij}^y, \tag{20}$$

where $C_{ij}^{x,y,z} \equiv \langle \psi_0 | S_i^{x,y,z} S_j^{y,z} | \psi_0 \rangle$ and we have used the result [26] $\langle \psi_0 | S_i^z S_j^y | \psi_0 \rangle = \langle \psi_0 | S_i^y S_j^z | \psi_0 \rangle = 0$ for arbitrary λ. For $\gamma \neq 1$, both $\left\langle\!\left\langle S_i^x S_j^x \right\rangle\!\right\rangle_{\eta_+}$ and $\left\langle\!\left\langle S_i^y S_j^y \right\rangle\!\right\rangle_{\eta_+}$ are real for $\xi = \frac{n\pi}{2}$. In particular, for $\xi = n\pi$:

$$\left\langle\!\left\langle S_i^x S_j^x \right\rangle\!\right\rangle_{\eta_+} = \cosh^2(\ln \gamma)\, C_{ij}^z - \sinh^2(\ln \gamma)\, C_{ij}^y,$$

$$\left\langle\!\left\langle S_i^y S_j^y \right\rangle\!\right\rangle_{\eta_+} = - \sinh^2(\ln \gamma)\, C_{ij}^z + \cosh^2(\ln \gamma)\, C_{ij}^y, \tag{21}$$

and for $\xi = (2n+1)\frac{\pi}{2}$:

$$\left\langle\!\left\langle S_i^x S_j^x \right\rangle\!\right\rangle_{\eta_+} = - \sinh^2(\ln \gamma)\, C_{ij}^z + \cosh^2(\ln \gamma)\, C_{ij}^y,$$

$$\left\langle\!\left\langle S_i^y S_j^y \right\rangle\!\right\rangle_{\eta_+} = \cosh^2(\ln \gamma) C_{ij}^z - \sinh^2(\ln \gamma)\, C_{ij}^y. \tag{22}$$

The diagonal correlation functions explicitly depend on γ and reproduce the known results in the Hermitian limit $\gamma = 1$.

The off-diagonal two-point correlation functions have the following form:

$$\left\langle\!\left\langle S_i^x S_j^y \right\rangle\!\right\rangle_{\eta_+} = \frac{i}{2} \sinh 2z \left(C_{ij}^z - C_{ij}^y \right),$$

$$\left\langle\!\left\langle S_i^x S_j^z \right\rangle\!\right\rangle_{\eta_+} = 0,$$

$$\left\langle\!\left\langle S_i^y S_j^z \right\rangle\!\right\rangle_{\eta_+} = 0, \tag{23}$$

where we have used the result [26] $\langle\psi_0|S_i^x S_j^y|\psi_0\rangle = 0$ and $\langle\psi_0|S_i^z S_j^x|\psi_0\rangle = 0$ for arbitrary λ. It may be noted that $\langle\langle S_i^x S_j^y\rangle\rangle_{\eta_+}$ is complex for $\gamma \neq 1$ and arbitrary ξ. Both $\langle\langle S_i^x S_i^z\rangle\rangle_{\eta_+}$ and $\langle\langle S_i^y S_i^z\rangle\rangle_{\eta_+}$ vanish for arbitrary λ. Other n-point correlation functions with higher values of n may be calculated in the same way. Some of them may become complex for $\gamma \neq 1$.

One of the major criticisms of the above results could be that all the one- and two-point correlation functions are not real simultaneously for a fixed ξ and $\gamma \neq 1$. The reason could be traced to the fact that although S_i^z are Hermitian in the Hilbert space of H that is endowed with the metric η_+, the same is not true for the spin variables S_i^x and S_i^y. As a result, in general, different correlation functions involving S_i^x and S_i^y are complex.

It is worth mentioning here that a common problem in the study of pseudo-Hermitian and $\mathcal{PT}$ symmetric quantum physics is that although the entire energy eigen values of a non-Hermitian Hamiltonian may become real with unitary time evolution, expectation values of other physical quantities of interest may not be real. Thus, a complete description of non-Hermitian Hamiltonian is not imminent. A common understanding in this regard is that the metric η_+ is not unique and a more general metric in the Hilbert space of H may be found so that all the correlation functions are real along with the eigenvalues. Since, a generalized metric that gives a complete description of H is not guaranteed *a priori*, identification of a proper set of operators those are Hermitian with respect to η_+ may give rise to a complete description of H. General prescription in this regard is already known [2]. To this end, we introduce a new set of spin operators,

$$
\begin{aligned}
T_i^x &:= -S_i^z, \\
T_i^y &:= -iS_i^x \sinh z + S_i^y \cosh z, \\
T_i^z &:= S_i^x \cosh z + iS_i^y \sinh z,
\end{aligned}
\tag{24}
$$

which satisfy the standard $SU(2)$ algebra. It should be noted here that $T_i^{x,y}$ are not Hermitian in the sense of Dirac hermiticity, i.e. $\langle u, T_i^{x,y} v\rangle \neq \langle T_i^{x,y} u, v\rangle$. However, the operators $T_i^{x,y,z}$ are Hermitian in the Hilbert space of H with respect to the metric η_+. The Hamiltonian H can be rewritten as

$$
H = -\sum_{i=1}^{N} \left(J T_i^x T_{i+1}^x + \beta T_i^z \right).
\tag{25}
$$

The Hamiltonian H is Hermitian, i.e. $\langle u, \eta_+ H v\rangle = \langle Hu, \eta v\rangle$. Using the identities,

$$
(U\rho)\, T_i^{x,y,z}\, (U\rho)^{-1} = S_i^{x,y,z},
\tag{26}
$$

and equation (13), it is easy to see that the n-point correlation functions of H and $\mathcal{H}$ are now identical,

$$
\langle\langle T_{i_1}^p T_{i_2}^q \cdots T_{i_n}^r\rangle\rangle_{\eta_+} = \langle\psi_m|S_{i_1}^p S_{i_2}^q \cdots S_{i_3}^r|\psi_m\rangle,
\tag{27}
$$

where the superscripts p, q, r can be identified with x, y, z. This implies that the Hamiltonian H that is non-Hermitian with respect to the condition of Dirac-hermiticity has in fact a consistent and complete description in terms of the new spin operators $T_i^{x,y,z}$ those are Hermitian in the Hilbert space of H that is endowed with the metric η_+. Moreover, energy eigenvalues and different correlation functions of H and $\mathcal{H}$ are identical.

One pertinent question that seems unavoidable at this point is whether the particular choice of the positive-definite metric η_+ has any role in establishing identical n-point correlation functions for H and $\mathcal{H}$. It seems that the answer is in the negative as long as the proper identification of the new set of spin operators is made for a given positive-definite metric. In

particular, for a given positive-definite metric $\eta_+ := \Gamma^2$, the Hermitian spin operators $\Sigma_i^{x,y,z}$ should be chosen as

$$\Sigma_i^{x,y,z} := \Gamma^{-1} S_i^{x,y,z} \Gamma, \qquad \forall\, i, \tag{28}$$

which would automatically imply identical correlation functions for H and $\mathcal{H}$. This observation is important, since it gives a metric independent description.

Finally, a few comments are in order:

 (i) The transverse-field Ising model is known to possess a global phase flip symmetry. The same symmetry is present in the pseudo-Hermitian Hamiltonian H also with the phase flip operator K given by

$$K := \prod_{i=1}^{N} T_i^z. \tag{29}$$

The operator K acts on $S_i^{x,y,z}$ in the following way:

$$K S_i^z K^{-1} = -S_i^z, \qquad K S_i^{\pm} K^{-1} = \mathrm{e}^{\mp 2z} S_i^{\mp}. \tag{30}$$

The phase flip symmetry acts quite non-trivially on $S_i^{x,y}$.

The Krammers–Wannier duality of the standard transverse-field Ising model can also be established for H. Defining a set of spin operators which obey $SU(2)$ algebra and are Hermitian with respect to the modified inner product in the Hilbert space,

$$\tau_i^x := \prod_{k \leqslant i} T_k^z, \qquad \tau_i^y := -T_i^y T_{i+1}^x \prod_{k \leqslant i-1} T_k^z, \qquad \tau_i^z := T_i^x T_{i+1}^x, \tag{31}$$

the Hamiltonian H can be rewritten as

$$H = -\sum_{i=1}^{N} \left(J \tau_i^z + \beta \tau_i^x \tau_{i+1}^x \right). \tag{32}$$

Based on the standard arguments, $\lambda = 1$ is determined as the critical point/line.

 (ii) The transverse Ising model undergoes quantum phase transition. Since $\mathcal{H}$ is related to H through the similarity transformation, H also undergoes quantum phase transition. The quantum critical line of H is determined by $\lambda = 1$ and it contains the quantum critical point of $\mathcal{H}$.

Near the critical line/point of the quantum phase transition, the equal-time correlations of the order parameter do not change for H and $\mathcal{H}$. Here we recall that the longitudinal correlation functions are identical for both the non-Hermitian and the Hermitian Ising models. However, the degree of quantum coherence among spins $S_i^{x,y,z}$ should be different for H and $\mathcal{H}$ due to the difference in the transverse correlation functions. The description of H in terms of the spin operators $S_i^{x,y,z}$ is incomplete and improper, since the correlation functions are not always real. A consistent and complete description in terms of the spin operators $T_i^{x,y,z}$ is possible and both transverse and longitudinal correlations functions are identical for H and $\mathcal{H}$. Consequently, the order parameter and the degree of quantum coherence near the critical point/line remain the same for both the Hamiltonians.

 (iii) The Hamiltonian H is quasi-Hermitian. A concept related to quasi-hermiticity is pseudo-hermiticity. One can show that H is pseudo-Hermitian, i.e., $H^\dagger = \theta H \theta^{-1}$, where $\theta := \rho^2$. The operator θ and its inverse are evaluated as given below

$$\theta = \prod_{i=1}^{N} \theta_i = \prod_{i=1}^{N} (\gamma^{-1} S_i^+ S_i^- + \gamma S_i^- S_i^+),$$

$$\theta^{-1} = \prod_{i=1}^{N} \theta_i^{-1} = \prod_{i=1}^{N} (\gamma S_i^+ S_i^- + \gamma^{-1} S_i^- S_i^+). \tag{33}$$

Note that $[\theta_i, \theta_j] = 0$ for $i \neq j$. Hence, the ordering of θ_is are not required in the expression above.

(iv) There are many physically motivated generalizations of the transverse Ising model [12, 13] and are important in the study of phase transitions. A non-Hermitian deformation of such models can be shown to be quasi-Hermitian. For example, consider the non-Hermitian Hamiltonian,

$$
H_1 = -\sum_{i=1}^{N} \left(J_i S_i^z S_{i+1}^z + K_i S_i^z S_{i+k}^z \right)
$$
$$
\qquad - \sum_{i=1}^{N} (\epsilon_{1,i} e^{i\xi_i} S_i^+ + \epsilon_{2,i} e^{-i\xi_i} S_i^-), \tag{34}
$$

where $J_i, K_i, \xi_i, \epsilon_{1,i}, \epsilon_{2,i}$ are real and k is an integer satisfying $1 < k < N$. The Hamiltonian H_1 is Hermitian for $\epsilon_{1,i} = \epsilon_{2,i}, \forall\, i$. Define the similarity operator ρ_1 and its inverse as

$$
\rho_1 = \prod_{i=1}^{N} \left(\gamma_i^{-\frac{1}{2}} S_i^+ S_i^- + \gamma_i^{\frac{1}{2}} S_i^- S_i^+ \right),
$$
$$
\rho_1^{-1} = \prod_{i=1}^{N} \left(\gamma_i^{\frac{1}{2}} S_i^+ S_i^- + \gamma_i^{-\frac{1}{2}} S_i^- S_i^+ \right), \tag{35}
$$

where $\gamma_i \equiv \sqrt{|\frac{\epsilon_{1,i}}{\epsilon_{2,i}}|}$. The non-Hermitian H_1 can be mapped to a Hermitian $\mathcal{H}_1$ through the similarity transformation,

$$
\mathcal{H}_1 = \rho_1 H_1 \rho_1^{-1}
$$
$$
\quad = -\sum_{i=1}^{N} \left(J_i S_i^z S_{i+1}^z + K_i S_i^z S_{i+k}^z \right)
$$
$$
\quad - \sum_{i=1}^{N} [\beta_i (e^{i\xi_i} S_i^+ + e^{-i\xi_i} S_i^-)], \tag{36}
$$

where $\beta_i \equiv \sqrt{|\epsilon_{1,i} \epsilon_{2,i}|}$. Thus, H is quasi-Hermitian.

We have considered a non-Hermitian deformation of the transverse Ising model that is also quasi-Hermitian. The transverse Ising model has been obtained from the starting non-Hermitian Hamiltonian through a similarity transformation. Consequently, both the models have identical eigen spectra, although the eigenfunctions are different. The metric in the Hilbert space, which makes the non-Hermitian model unitary and ensures a complete set of states, has been constructed explicitly. Although the longitudinal correlation functions are identical for both the non-Hermitian and the Hermitian Ising models, the difference shows up in the transverse correlation functions, which have been calculated explicitly. However, the transverse correlation functions are not always real. In order to give a complete and consistent description, we have identified a proper set of Hermitian spin operators in the Hilbert space of the non-Hermitian Hamiltonian in terms of which all the correlation functions of the non-Hermitian Hamiltonian become real and identical to that of the standard transverse Ising model. The non-Hermitian Hamiltonian undergoes quantum phase transitions and it is expected that around the quantum critical line both the order parameter and the degree of quantum coherence of spins should be identical to that of the standard transverse Ising model.

Acknowledgments

References

[1] Bender C M 2005 arXiv:quant-ph/0501052
    Bender C M, Brody D C and Jones H F 2003 *Am. J. Phys.* **71** 1095
    Bender C M and Boettcher S 1998 *Phys. Rev. Lett.* **80** 5243
    Bender C M and Boettcher S 1998 *J. Phys. A: Math. Gen.* **31** L273
    Bender C M, Boettcher S and Meisinger P N 1999 *J. Math. Phys.* **40** 2210
[2] Mostafazadeh A 2008 arXiv:0810.5643
    Mostafazadeh A 2002 *J. Math Phys.* **43** 205
    Mostafazadeh A 2002 *J. Math Phys.* **43** 2814
    Mostafazadeh A 2002 *J. Math Phys.* **43** 3944
    Mostafazadeh A and Batal A 2004 *J. Phys. A: Math. Gen.* **37** 11645
    Mostafazadeh A 2002 *Nucl. Phys.* B **640** 419
[3] Scholtz F G, Geyer H B and Hahne F J W 1992 *Ann. Phys.* **213** 74
[4] Dorey P, Dunning C and Tateo R 2001 *J. Phys. A: Math. Gen.* **34** 5679
    Dorey P, Dunning C and Tateo R 2007 *J. Phys. A: Math. Theor.* **40** R205
[5] Ghosh P K and Gupta K S 2004 *Phys. Lett.* A **323** 29
[6] Swanson M S 2004 *J. Math. Phys.* **45** 585
[7] Quesne C 2007 *J. Phys. A: Math. Theor.* **40** F745
    Musumbu D P, Geyer H B and Heiss W D 2007 *J. Phys. A: Math. Theor.* **40** F75
    Sinha A and Roy P 2007 *J. Phys. A: Math. Theor.* **40** 10599
[8] Ghosh P K 2005 *J. Phys. A: Math. Gen.* **38** 7313
    Deguchi T and Ghosh P K 2009 *Phys. Rev.* E **80** 021107
    Deguchi T, Ghosh P K and Kudo K 2009 *Phys. Rev.* E **80** 026213
[9] Korff C 2008 arXiv:0803.4500 [math-ph]
    Korff C and Weston R A 2007 *J. Phys. A: Math. Theor.* **40** 8845
[10] Jin L and Song Z 2009 arXiv:0904.1513
[11] Sachdev S 1999 *Quantum Phase Transitions* (Cambridge: Cambridge University Press)
[12] Chakrabarti B K, Dutta A and Sen P 1996 *Quantum Ising Phases and Transitions in Transverse Ising Models* (Heidelberg: Springer)
[13] Dziarmaga J 2005 *Phys. Rev. Lett.* **95** 245701
    Dziarmaga J and Rams M M arXiv:0904.0115
[14] Wu L-A *et al* 2004 *Phys. Rev. Lett.* **93** 250404
    de Oliveira T R *et al* 2006 *Phys. Rev. Lett.* **97** 170401
    Chen H D 2006 arXiv:cond-mat/0606126
[15] McCoy B M and Wu T T 1968 *Il Nuovo Cimento* B **56** 311
[16] Au-Yang H and Perk J H H 1989 *Adv. Stud. Pure Math.* **19** 57
[17] Albertini G, McCoy B M and Perk J H H 1989 *Adv. Stud. Pure Math.* **19** 1
[18] Alcaraz F C, Droz M, Henkel M and Rittenberg V 1994 *Ann. Phys., NY* **230** 250
    Derrida B, Evans M R, Hakim V and Pasquier V 1993 *J. Phys. A: Math. Gen.* **26** 1493
    Sasamoto T and Wadati M 1998 *J. Phys. A: Math. Gen.* **31** 6057
    Schütz G M 2000 Integrable stochastic many-body systems *Phase Transition and Critical Phenomena* vol 19, ed ed C Domb and J L Lebowitz (London: Academic)
[19] Ghosh P K 2009 arXiv:0908.321
[20] von Gehlen G 1991 *J. Phys. A: Math. Gen.* **24** 5371
    von Gehlen G 1994 *Int. J. Mod. Phys.* B **8** 3507
[21] Yang C N and Lee T D 1952 *Phys. Rev.* **87** 404
    Lee T D and Yang C N 1952 *Phys. Rev.* **87** 410
[22] Cardy J L 1985 *Phys. Rev. Lett.* **54** 1354
[23] Castro-Alvaredo O A and Fring A 2009 arXiv:0906.4070

[24] Lieb E, Schultz T and Mattis D 1961 *Ann. Phys.* **16** 407
[25] Pfeuty P 1970 *Ann. Phys.* **57** 79
[26] McCoy B M 1968 *Phys. Rev.* **173** 531
      Barouch E and McCoy B M 1971 *Phys. Rev.* A **3** 786
      Johnson J D and McCoy B M 1971 *Phys. Rev.* A **4** 2314
[27] Perk J H H and Au-Yang H 2009 *J. Stat. Phys.* **135** 599